3) Conditional Probability & Independence

Knowing that an event occured forces us to reasses the probability of another outcome.

Example 3.1: Consider a game of rolling a fair die twice.

- If total score higher than 9: win £6
- Otherwise: lose £1.

The probability of winning is probability of getting

(4,6), (6,4), (5,5), (5,6), (6,5), (6,6) i.e.

the probability is $\frac{6}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$

Now suppose the first roll of die gives you 6.

The number of possible scores has been reduced from 6 to 36.

The original sample space contains:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

→ Since the first outcome is 6, the sample space is <u>reduced to contain elements that has 6 as its first element</u>.

<u>After die stops at 6</u>, the new ~~spample~~ sample space becomes the last row:

New sample space:

$$\Omega = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

Let A: be the event of winning the bet.

B: be the event of first die stopping at 6.

Before extra information:

$$P(A) = 1/6 \qquad P(B) = 1/6$$

$$P(A \cap B) = P(\{(6,4),(6,5),(6,6)\})$$

$$= 3/36$$

After additional info we have a new probability call it P_B such that

$$P_B(A) = 1/2$$

$$P_B(B) = 1$$

Note that:

$$\frac{P(A \cap B)}{P(B)} = \frac{3/36}{1/6} = 1/2 \, , \quad \frac{P(B \cap B)}{P(B)} = \frac{1/6}{1/6} = 1$$

i.e.

apply to original sample space

$$\boxed{P_B(A) = \frac{P(A \cap B)}{P(B)}} \text{ and } \boxed{P_B(B) = \frac{P(B \cap B)}{P(B)}}$$

Defn 3.2: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let B be any event such that $P(B) > 0$.

For any event A, the conditional probability of A given B is

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

("call B the conditioning")

- Intuition of the above formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$:

Basically B has occured and we want probability of A occuring given B has occured. You are basically narrowing down your "pool" of possible outcomes to events such that B has occured. When we know B occured, the occurences of A are all and exactly those situations where both A and B occur, and since you are assuming B occured. The total number of possibilities are reduced to only those where B happened.
Hence

$$P(A|B) = \frac{\#\text{ of occurences of } A \text{ and } B}{\#\text{ of occurences where } B \text{ occured}} = \frac{P(A \cap B)}{P(B)}$$

We prove that the conditional probability also satisfies the axioms of probability.

Theorem 3.2: For any event B with $P(B) > 0$, the function P_B defined by

$$P_B(A) = P(A|B)$$

for any event A is a probability function $\mathcal{F}$ so that $(\Omega, \mathcal{F}, P_B)$ is a probability space

proof: Need to show that P_B satisfies (P1), (P2), (P3), using that P does.

(P1):

By defn from thm, $P_B(A) > 0$ as $P_B(A) = \frac{P(B \cap A)}{P(B)}$ and $P(A \cap B) \geq 0$ and $P(B) > 0$.

Since $A \cap B \subseteq B$, by (P7) $P(A \cap B) \leq P(B)$. So

$$0 \leq \frac{P(A \cap B)}{P(B)} \leq \frac{P(B)}{P(B)} = 1$$

$\Rightarrow$

$$0 \leq P_B(A) \leq 1 \Rightarrow P_B(A) \in [0,1].$$

(P2):

$$P_B(\Omega) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(P3):

If $\{E_i \mid i \in I\}$ is a countable collection disjoint events, then the collection $\{E_i \cap B \mid i \in I\}$ is also disjoint. Thus using (P3) for probability, P,

explained using venn diagram next page

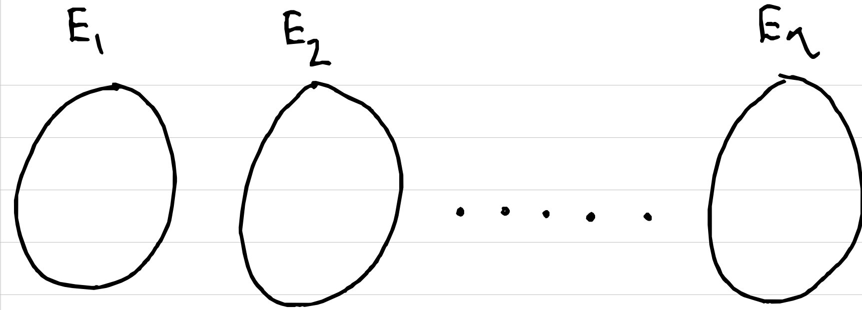
$$P_B\left(\bigcup_{i \in I} E_i\right) = \frac{P\left(\left(\bigcup_{i\in I} E_i\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\left(\bigcup_{i\in I} E_i \cap B\right)\right)}{P(B)}$$

$$= \sum_{i \in I} \frac{P(E_i \cap B)}{P(B)}$$

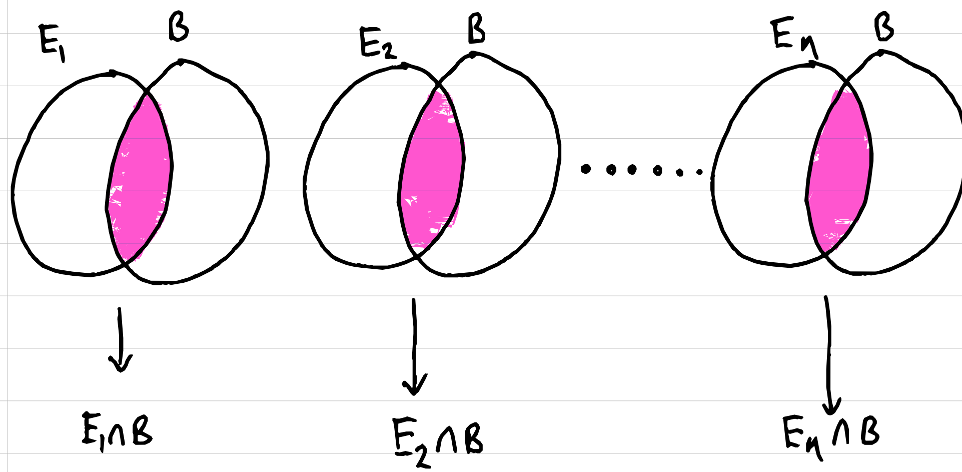
$$= \sum_{i \in I} P_B(E_i)$$

$\blacksquare$



are all disjoint.

So



$(E_1 \cap B), (E_2 \cap B) \ldots, (E_n \cap B)$ are all disjoint.

Example 3.4 Game involves tossing two fair coins:

- player wins by getting atleast one tail.

Given Peter played the game and won, what is the probability he got 2 tails.

the conditioning

Solution: At start of game, sample space is

$$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}.$$

Let B be event that Peter wins the game:

conditioning event.

$$B = \{(H,T), (T,H), (T,T)\}$$

Let A be the event that he gets 2 tails.

$$A = \{(T,T)\}$$

$P(B) = 3/4$, $P(A) = 1/4$, $P(A \cap B) = 1/4$.

Conditioning on event B, we see that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

3.2 The Multiplication rule:

Example: 3.5 Consider drawing 2 balls out of a bag (without replacement)

Bag contains
- 8 white balls
- 4 red balls.

what is the probability both are red balls:

Solution: Let
- R_1 be the event that the first ball is red.
- R_2 be the event that the second ball is red.

The example question asks for $P(R_1 \cap R_2)$.

For the first draw, $P(R_1) = 1/3$ $\left(\frac{4 \text{ red}}{12 \text{ total}}\right)$ is easy to determine. All balls are equally likely and one third of balls are red.

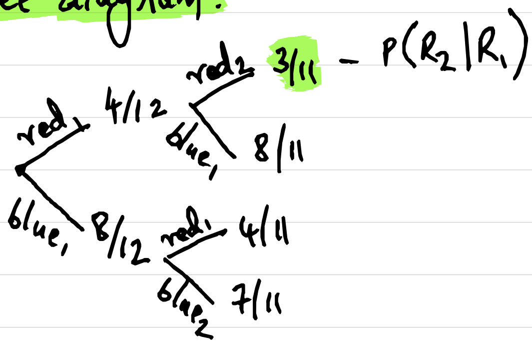
Less easy to determine $P(R_2)$ because we do not know whether by the time the second ball is picked, whether there are still 4 or 3 red balls in the bag. That depends on outcome of first draw.

We can however determine the conditional probability $P(R_2 | R_1)$ that the second ball is red given that first ball was red because given the information we already know, that there are 3 red balls left and there are 11 balls left altogether,

$$P(R_2 | R_1) = 3/11.$$

So due to conditioning the total is know 11 balls, i.e. where red ball was picked and there are 3 red balls Hence $P(R_2 | R_1) = 3/11$.

Using tree diagram:



Now recall defn of conditional probability:

$$P(R_2 | R_1) = \frac{P(R_2 \cap R_1)}{P(R_1)}$$

we solve that for $P(R_2 \cap R_1)$ giving

$$P(R_2 \cap R_1) = P(R_2 | R_1) . P(R_1)$$

$$= \frac{\cancel{3}}{11} . \frac{1}{\cancel{3}}$$

$$= \frac{1}{11}$$

Theorem: 3.6 (Multiplication rule):
Let A, B be events with $P(B) > 0$ Then

$$\boxed{P(A \cap B) = P(A|B) . P(B)}$$

proof: $$P(A \cap B) = \frac{P(A \cap B)}{P(B)} . P(B) = P(A|B) . P(B).$$

$\blacksquare$

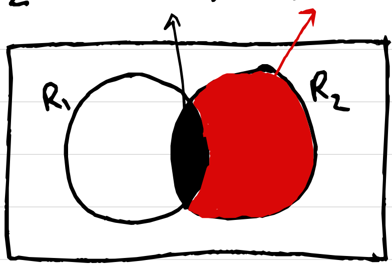
Example 3.5 continued: We now address the question:

what is $p(R_2)$.

We use the trick of splitting up event R_2 into union of disjoint events.

Because

$$R_2 = (R_2 \cap R_1) \cup (R_2 \cap R_1^c)$$



and

$$(R_2 \cap R_1) \cap (R_2 \cap R_1^c) = \emptyset,$$

we can write

$$p(R_2) = p((R_2 \cap R_1) \cup (R_2 \cap R_1^c))$$

(by (P3)) $= p(R_2 \cap R_1) + p(R_2 \cap R_1^c)$

(by Thm 3.6) $= p(R_2 | R_1) p(R_1) + p(R_2 | R_1^c) \cdot p(R_1^c)$

$$P(R_2) = P\big((R_2 \cap R_1) \cup (R_2 \cap R_1^c)\big)$$

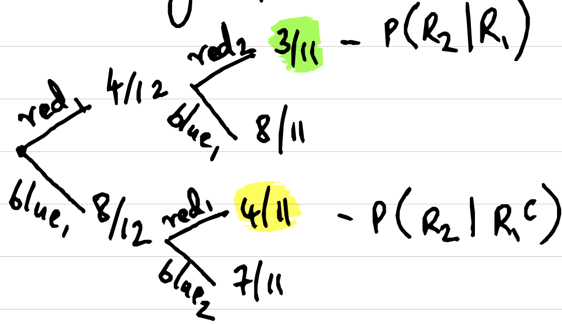
$$(\text{by } (P3)) \quad = P(R_2 \cap R_1) + P(R_2 \cap R_1^c)$$

$$(\text{by Thm } 3.6) \quad = P(R_2 | R_1) P(R_1) + P(R_2 | R_1^c) \cdot P(R_1^c)$$

All probabilities on the right-hand side are easy to calculate.

$P(R_2 | R_1^c) = 4/11$ as if first ball was not red, there are still 4 red balls amongst the 11 balls in the bag.

Also use tree diagram:



And $P(R_1^c) = 1 - P(R_1) = 2/3$

Substituting into above equation gives:

$$P(R_2) = \frac{3}{11} \cdot \frac{1}{3} + \frac{4}{11} \cdot \frac{2}{3} = \frac{1}{3}$$

Example 3.7: Assume 60% (0.6) of consumers have mobile phone from manufacturer A,
30% (0.3) from manufacturer B.
10% (0.1) from manufacturer C.

The probability that phone from manufacturer A spontaneously go up in flames is 1%. (0.01)
For B it is 2%. (0.02)
For C it is 3%. (0.03)

Probability that random customer phone go up in flames?

Solution: Event M_A: random customer has phone from manufacturer A.

Event M_B: random customer has phone from manufacturer B

Event M_C: random customer has phone from manufacturer C.

Note that events M_A, M_B, M_C are disjoint.

$$M_A \cap M_B = M_B \cap M_C = M_A \cap M_C = \emptyset$$

Since $M_A + M_B + M_C = 60\% + 30\% + 10\% = 100\%$

$$\Omega = M_A \cup M_B \cup M_C$$

Event F: customers phone goes into flames.

$$P(M_A) = 0.6 \quad P(M_B) = 0.3 \quad P(M_C) = 0.1$$

and

$$P(F|M_A) = 0.01 \quad P(F|M_B) = 0.02 \quad P(F|M_C) = 0.03$$

Now split F whose probability we want to calculate: into 3 disjoint events:

$$F = (F \cap M_A) \cup (F \cap M_B) \cup (F \cap M_C)$$

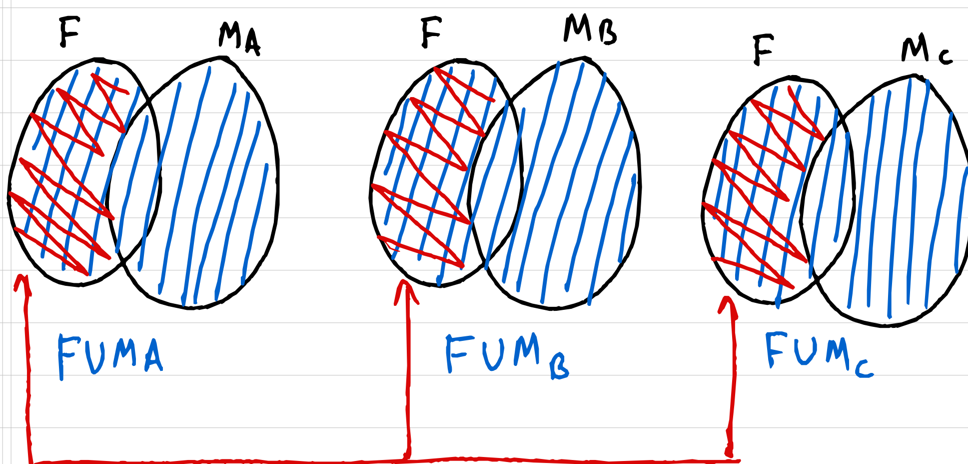
$$F = (F \cap M_A) \cup (F \cap M_B) \cup (F \cap M_C)$$

$$= F \cap (M_A \cup M_B \cup M_c) \qquad \text{(distributive law)}$$

$$= F \cap \Omega \qquad \text{(disjoint events)}$$

$$= F$$

Venn diagram:



Common elements

$$= (F \cap M_A{}^c) \cup (F \cap M_B{}^c) \cup (F \cap M_c{}^c)$$

$$= F \cap (M_A^c \cup M_B^c \cup M_c^c)$$

$$= F \cap (M_A \cap M_B \cap M_c)^c \qquad \text{(de Morgan's law)}$$

$$= F \cap \Omega = F$$

Therefore by Axiom (P3)

$$P(F) = P((F\cap M_A) \cup (F\cap M_B) \cup (F\cap M_C))$$

$$= P(F\cap M_A) + P(F\cap M_B) + P(F\cap M_C)$$

$$= P(F|M_A)P(M_A) + P(F|M_B)P(M_B)$$

$$+ P(F|M_C) . P(M_C)$$

$$= (0.01)(0.6) + (0.02).(0.3) + (0.03).(0.1)$$

$$= 0.015$$

So the probability that the phone of random customer phone going to flames is 1.5%.

Note: Equation got a bit long for three alternative events.
We can shorten this using the set index notation in section 2.2.

index set $I = \{A, B, C\}$.
Rewrite equations in Example 3.7 as:

$$M_i \cap M_j = \phi \quad \forall\, i, j \in I,\ i \neq j$$

$$\Omega = \bigcup_{i \in I} M_i \qquad = (M_A \cup M_B \cup M_C)$$

$$F = \bigcup_{i \in I} (F \cap M_i) \qquad = \left[(F \cap M_A) \cup (F \cap M_B) \cup (F \cap M_C) \right]$$

and

$$P(F) = \sum_{i \in I} P(F \mid M_i)\, P(M_i)$$

We can now use this notation to formalise the idea of splitting the sample space into a union of disjoint sets.

Defn 3.8: We say that a countable collection of events

$$\{B_i \mid i \in I\}$$

is a partition of Ω if

$$B_i \cap B_j = \emptyset \quad \forall\, i, j \in I,\ i \neq j$$

and

$$\Omega = \bigcup_{i \in I} B_i$$

Theorem 3.9: (Law of Total Probability or Partition Theorem):

Let $\{B_i \mid i \in I\}$ be a partition of a sample space Ω such that $P(B_i) > 0$ for each $i \in I$. Then:

$$P(A) = \sum_{i \in I} P(A \mid B_i) P(B_i) \quad \forall \text{ events } A.$$

proof: Since $P(\bigcup_{i \in I} B_i) = P(\Omega) = 1$, for each event A,

$$P(A) = P\Big(\underbrace{A \cap \big(\bigcup_{i \in I} B_i\big)}_{P(A \cap \Omega) = P(A)}\Big) = P\Big(\underbrace{\bigcup_{i \in I} (A \cap B_i)}\Big)$$

$$= \sum_{i \in I} P(A \cap B_i) \quad (*_1)$$

$$= \sum_{i \in I} P(A \mid B_i) P(B_i)$$

distributive law:

$(A \cap [B_1 \cup B_2 \cup \dots]) = (A \cap B_1) \cup (A \cup B_1) \dots$

$= \bigcup_{i \in I} (A \cap B_i)$

$(\star_1)$

$$P\left(\bigcup_{i\in I}(A\cap B_i)\right) = \sum_{i\in I} P(A\cap B_i)$$

let $i\in I = \mathbb{N}$

$$P\big((A\cap B_1)\cup(A\cap B_2)\cup(A\cap B_3)\cup\dots\cup(A\cap B_n)\big)$$

Since all B_1 to B_n are disjoint,
$(A_1\cap B_1)$ to $(A\cap B_n)$ are all disjoint.
(shown before) (in venn diagram)

$=$

$$P(A\cap B_1) + P(A\cap B_2) + \dots + P(A\cap B_n)$$

$=$

$$\sum_{i\in I} P(A\cap B_i)$$

Example: 3.10 On thursday night, stay in or go to fibbers.

Probability of going to fibbers is $\frac{4}{5}$.

If you choose to stay in, ↗ conditioning probability of staying awake in calculus lecture is 9/10

If you choose to go, probability of staying awake in calculus lecture is 1/5

what is probability of staying awake during calculus lecture on friday:

Solution: A: Event you stay awake

F: Event going to fibbers.

$P(F) = 4/5$. $P(F^c) = 1/5$ (partition) $F \cap F^c = \phi$, $F \cup F^c = \Omega$

$P(A \mid F^c) = 9/10$ $P(A \mid F) = 1/5$

By law of total probability

$$P(A) = P(A \mid F)\, P(F) + P(A \mid F^c)\, P(F^c)$$

$$= \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{9}{10}\right)\left(\frac{1}{5}\right) = \frac{17}{40} = 0.34$$

Example 3.11: (Monty hall problem):
In a particular game show: you have to choose one of 3 doors.

- Behind one door, there is a car.
- other 2 doors there is a goat.

Once you have chosen a door, and before it is opened, the game show host opens another door which he knows there is a goat.
You are then given an opportunity to revise your choice.
Should you swap to the unopened door?

Two possible arguments are as follows:

1) after door has opened, the probability that the car is behind each of the remaining closed doors is $1/2$, so it does not matter whether you swap or not.

2) At start of game you had probability of $1/3$ of choosing right door. If you don't swap, nothing has changed, chance of winning is $1/3$, so you should swap.

Here argument 2) is the correct one. It can be shown using law of total probability.

Solution: Let
- R: be the event "prize behind door you initially chose"
- S: be the event you win prize by switching doors.

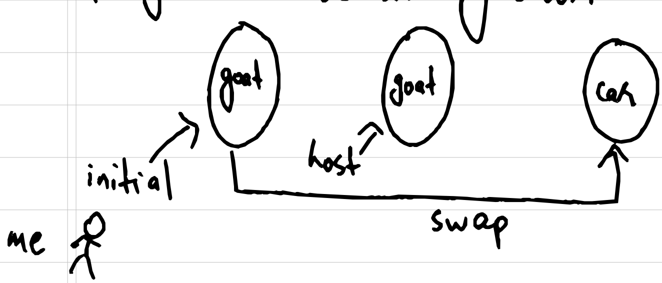
If you choose the right door, switching will make you lose, so

$$\boxed{P(S|R) = 0}$$

If you chose wrong door, switching will make you win for sure,

$$\boxed{P(S|R^c) = 1}$$

Explanation of $P(S|R^c)$:
if you choose wrong door



host opens door with a goat. The only door left is one with car, so you can only swap to the door with the car, so probability of getting a car after choosing wrong door is 1.

Calculating probability of winning by swapping doors by using law of total probability:

$$P(S) = P(S|R)P(R) + P(S|R^c)P(R^c)$$

$$= 0 \cdot 1/3 + 1 \cdot 2/3$$

$$= 2/3$$

So you should switch doors!

Example 3.10 continued: what is the probability of having gone to fibbers if you stayed awake during lecture.

Solution: Need to calculate $P(F|A)$.

Using defn 3.2, of conditional probability, multiplication rule (Thm 3.6) and law of total probability (Thm 3.9):

$$P(F|A) \overset{\text{defn 3.2}}{=} \frac{P(F \cap A)}{P(A)} \overset{\text{Thm 3.6 Multiplication rule}}{=} \frac{P(A|F)\,P(F)}{P(A)}$$

(law of total probability, Thm 3.9)

$$= \frac{P(A|F)\,P(F)}{P(A|F)\,P(F) + P(A|F^c)\,P(F^c)}$$

partition set is $\{F, F^c\}$

$$= \frac{1/5 \cdot 4/5}{17/50} = \frac{8}{17}$$

Example 3.7 continued: Introduce events

- M_i = phone manufactured by manufacturer i

$$i \in \{A, B, C\}$$

- F = phone goes up in flames.

What is conditional probability that a phone was made by a particular manufacturer if it goes up in flames.

↳ conditioning

Solution: Using law of total probability, with a partition containing M_A M_B M_C

$$P(M_A \mid F) \overset{\text{defn } 3.2}{=} \frac{P(M_A \cap F)}{P(F)} \overset{\text{Thm 3.6 multiplication rule}}{=} \frac{P(F \mid M_A)\,P(M_A)}{P(F)}$$

$$= \frac{P(F \mid M_A)\,P(M_A)}{P(F \mid M_A)\,P(M_A) + P(F \mid M_B)\,P(M_B) + P(F \mid M_C)\,P(M_C)}$$

↳ law of total probability Thm 3.9

Partition set $\{M_A, M_B, M_C\}$

$$= \frac{(0.01)(0.6)}{0.015} = \frac{6}{15} = \frac{2}{5} = 0.4$$

Theorem 3.12: (Baye's theorem) (used to swap conditional probability)

Let $\{B_i \mid i \in I\}$ be a partition of sample space Ω. such that $P(B_i) > 0$ for all $i \in I$.

If A is an event with $P(A) > 0$ then

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A)} \qquad \forall i \in I$$

$$= \frac{P(A \mid B_i)\, P(B_i)}{\sum_{j \in I} P(A \mid B_j)\, P(B_j)} \qquad \forall i \in I$$

proof:

Defn 3.2 — Thm 3.6 multiplication rule

$$P(B_i \mid A) \overset{\text{Defn 3.2}}{=} \frac{P(B_i \cap A)}{P(A)} \overset{\text{Thm 3.6}}{=} \frac{P(A \mid B_i)\, P(B_i)}{P(A)}$$

Thm 3.9 law of total probability.

$$= \frac{P(A \mid B_i)\, P(B_i)}{\sum_{j \in I} P(A \mid B_j)\, P(B_j)}$$

$\blacksquare$

From Theorem 3.12, the following corollary can be deduced:

Corollary 3.13 (Bayes theorem for 2 alternative):

For any event A and B, with positive probability,

$$p(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$$

$$= \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)}$$

proof Use the partition $\{B, B^c\}$ in Theorem 3.12.

Example 3.14: Suppose that a cancer diagnostic test is 95% accurate both on those who do have cancer and on those who don't. ⟷ conditioning

Assume that 0.5% (0.005) of population have cancer, what is probability that a particular individual has the disease, given that the test is positive

Solution: Let C: be the event that the individual has cancer

T: be the event that test is positive

Given in question:

$$P(T \mid C) = 0.95 = 1 - P(T^c \mid C)$$

$$P(T \mid C^c) = 0.95$$

$$P(C) = 0.005 \Rightarrow P(C^c) = 1 - 0.005 = 0.995.$$

$$P(C \mid T) = \frac{P(T \mid C) \cdot P(C)}{P(T \mid C) \cdot P(C) + P(T \mid C^c) P(C^c)}$$

$$= \frac{(0.95) \cdot (0.005)}{(0.95)(0.005) + (0.05)(0.995)}$$

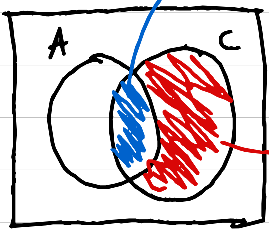
$$\approx 0.087.$$

So in spit of high accuracy, the probability that the patient has cancer is less than 9%.

Lemma: $\boxed{P(A|C) + P(A^c|C) = 1}$

proof: $P(A|C) + P(A^c|C) = \dfrac{P(A\cap C)}{P(C)} + \dfrac{P(A\cap C^c)}{P(C)}$

$$= \frac{P(A\cap C) + P(A\cap C^c)}{P(C)}$$



$(A\cap C)\cap(A\cap C^c) = \phi$

$$= \frac{P((A\cap C)\cup(A\cap C^c))}{P(C)}$$

$$= \frac{P(C)}{P(C)} = 1$$

$(A\cap C)\cup(A\cap C^c) = C$

$\blacksquare$

3.4 Independance

Situations where the knowledge that event B occurs does not influence the probability of an event A.

Defn 3.15: Events A and B are independant if

$$P(A \cap B) = P(A) . P(B)$$

Theorem 3.16: Let A and B be events with $P(B) > 0$. The following statements are equivalent:

1) A and B are independant

2) $P(A|B) = P(A)$

↳ if B occurs, it has no effect on A since they are independant.

proof: To show 1 and 2 are equivalent, we need to show

$1 \Rightarrow 2$ and $2 \Rightarrow 1$.

$(1 \Rightarrow 2)$:

Using definition of conditional probability:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A).\cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

$\hookrightarrow$ by 1 and 3.15.

$(2 \Rightarrow 1)$

Using multiplication rule:

$$P(A\cap B) = P(A|B).P(B)$$

$$= P(A).P(B) \quad \text{by } 2.$$

$\blacksquare$

Example 3.17: A playing card picked random from an ordinary deck of cards (52 cards).

Events: $A = \{\text{card is red}\}$

$B = \{\text{card is an eight}\}$

A and B are independant since

$$P(A \cap B) = P(\text{card is red and } 8)$$

$$= \frac{2}{52} = \left(\frac{26}{52}\right)\left(\frac{4}{52}\right) = P(A)\,P(B).$$

Theorem 3.18: If A and B are independant, so are A and B^c are independant.

We have $A = (A \cap B) \cup (A \cap B^c)$ $(A \cap B) \cap (A \cap B^c) = \emptyset$

Thus

$$P(A) = P(A \cap B) + P(A \cap B^c) \quad \text{by } (P3).$$

Solving for $P(A \cap B^c)$

$$P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$
$$= P(A)(1 - P(B))$$

and by (P4),

$$P(A \cap B^c) = P(A) . P(B^c).$$

$\blacksquare$

Defn 3.19: A collection $\mathcal{A} = \{A_i \mid i \in I\}$ of events is called independent if for all finite subsets J of I,

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_j)$$

Note: $\boxed{A \text{ independant of } B \iff B \text{ independant of } A}$

Clearly defn 3.15 is a special case of defn 3.19, when $|I| = 2$.

If $|I| = 3$, eg. $\mathcal{A} = \{A_1, A_2, A_3\}$, then the events A_1, A_2, A_3 are independant <u>if and only if</u> <u>all</u> of the following inequalities hold:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) . P(A_2) . P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1) . P(A_2)$$

$$P(A_1 \cap A_3) = P(A_3) \, P(A_1)$$

$$P(A_2 \cap A_3) = P(A_2) . P(A_3).$$

<u>Note:</u> <u>Pairwise independance not enough to guarantee independance of 3 events</u>

The following example illustrates this:

Example 3.20: A family has 3 children, each of which, it is equally likely to have a boy or girl.

Define events:

$A = \{\text{all children are of same sex}\}$

$B = \{\text{at most one boy}\}$ (≤ 1)

$C = \{\text{the family includes boy and girl}\}$

Show that A and B are independant.
B and C are independant

Is A independant of C?

Do the above results hold if boy and girl are not born with equal probability.

Solution Sample space consists of ordered tripples

$\Omega = \{b, g\} \times \{b, g\} \times \{b, g\} = \{(i, j, k) \mid i, j, k \in \{b, g\}\}$

b stands for boy   let $b = 1/2$ } equally
g stands for girl. $g = 1/2$ } likely.

$P(A) = P(\{(b,b,b)\} \cup \{(g,g,g)\})$

$= P(\{b,b,b\}) + P(\{g,g,g\})$

$= P(\text{a boy is born})^3 + P(\text{a girl is born})^3$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}$

where third equality followed by independance.

$P(B) = P(\{(g,g,g), (b,g,g), (g,b,g), (g,g,b)\})$

$= P(\{(g,g,g)\}) + P(\{(b,g,g)\}) + P(\{(g,b,g)\}) + P(\{(g,g,b)\})$

$= 4\left(\frac{1}{3}\right)^3 = \frac{1}{2}$

Since $C = A^c$, $P(C) = P(A^c)$
$$= 1 - P(A)$$
$$= 1 - \frac{1}{4} = 3/4.$$

Now,

$$P(A \cap B) = P(\{(g,g,g)\})$$
$$= \left(\frac{1}{2}\right)^3 = \frac{1}{8} = P(A) \cdot P(B)$$

So events <u>A and B are independant</u>.

$$P(B \cap C) = P(\{(b,g,g), (g,b,g), (g,g,b)\}$$
$$= 3\left(\frac{1}{2}\right)^3 = \frac{3}{8} = P(B) \cdot P(C)$$

So <u>events B and C are independant</u>.

However $A \cap C = \phi$ so $P(A \cap C) = 0 \neq P(A).P(C)$

So events A and C are not independant.

The conclusions drawn so far are very much dependant on the fact that boys and girls are equally likely to be born.

If $P(\text{a boy is born}) = p \neq \frac{1}{2}$

$P(\text{a girl is born}) = 1-p$

Then $P(A \cap B) = (1-p)^3$ whereas

$$P(A)P(B) = [p^3 + (1-p)^3][(1-p)^3 + 3p(1-p)^2]$$

So events are no longer independant unless either $p=0$, or $p=1$.

Note: We could have worked with smaller sample space $\Omega = \{0,1,2,3\}$, where numbers refer to numbers of boys among 3 children

In this notation, $A = \{0,3\}$ $B = \{0,1\}$, $C = \{1,2\}$

In this sample space, not all outcomes are equally probable.

Rather:

for $P(\{0\})$ one possible permutation: $\{(g,g,g)\}$

for $P(\{3\})$: $\{b,b,b\}$

$$P(\{0\}) = P(\{3\}) = \frac{1}{8}$$

$$P(\{1\}) = P(\{2\}) = \frac{3}{8}$$

3 possible permutations

for $\{1\}$, it is (b,g,g), (g,b,g), (g,g,b),

Hence $P(\{1\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Similar for $P(\{2\})$

Defn 3.21: Events A and B are <u>conditionally independant</u> given event C if

$$P((A\cap B)|C) = P(A|C).\,P(B|C)$$

It follows that theorem 3.16 also applies so that for example:
if $P(B|C)>0$ then the conditional independance of A and B given C implies that

$$P(A|(B\cap C)) = P(A|C)$$

<u>proof</u>:

$$P(A|(B\cap C)) = \frac{P(A\cap B\cap C)}{P(B\cap C)}$$

$$= \frac{P((A\cap B)|C)\,P(C)}{P(B\cap C)}$$

$$= \frac{P(A|C)\,P(B|C)\,P(C)}{P(B\cap C)} = \frac{P(A|C)\,P(\cancel{B\cap C})}{P(\cancel{B\cap C})}$$

$$= P(A|C)$$

■

Conditional Baye's theorem:

An alternative formulation to Bayes' theorem, which whilst being mathematically identical to theorem 3.12 is common enough and sufficiently different looking that it is worth writing down!

This is the "conditional" Bayes theorem where we have an extra "given" statement on both sides of the equation.

Before we get to the theorem though, it is worth noting how to chain together conditions

Theorem: Let A, B, C be events such that $P(C) > 0$ and $P(B \cap C) > 0$. Then

$$\boxed{P_c(A|B) = P(A|B \cap C)}$$

proof: Starting with the definition of conditional probability for P_c, we have

$$P_c(A|B) = \frac{P_c(A \cap B)}{P_c(B)}$$

Now

$$P_C(A \cap B) = P(A \cap B | C) \qquad \text{by defn}$$
$$= \frac{P(A \cap B \cap C)}{P(C)}$$

Similarly

$$P_C(B) = P(B|C) \qquad \text{by defn.}$$
$$= \frac{P(B \cap C)}{P(C)}$$

Therefore by substitution

$$P_C(A|B) = \frac{\dfrac{P(A \cap B \cap C)}{\cancel{P(C)}}}{\dfrac{P(B \cap C)}{\cancel{P(C)}}} \begin{matrix} \times P(C) \\ \times P(C) \end{matrix}$$

$$= \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A \cap (B \cap C))}{P(B \cap C)}$$

$$= P(A | B \cap C) \qquad \blacksquare$$

Corollary: (Conditional Baye's Theorem):

Let A, B and C be events such that $P(C)$, $P(A\cap C)$ and $P(B\cap C)$ are all 0. Then

$$\boxed{P(B|A\cap C) = \frac{P(A|B\cap C)\cdot P(B|C)}{P(A|C)}}$$

Proof:

$$P(B|A\cap C) = P_C(B|A) \quad \text{(by previous thm)}$$

$$= \frac{P_C(A|B)\cdot P_C(B)}{P_C(A)} \quad \text{(by Bayes thm)}$$

$$= \frac{P(A|B\cap C)\cdot P(B|C)}{P(A|C)} \quad \text{(by previous defn)}$$

$\blacksquare$

Example 3.11 revisited: We can use the conditional Baye's theorem for a more explicit treatement of monty hall problem.

Intuitively, one might want to phrase the problem as:

> "Given I chose door 1, and Monty revealed a goat behind door 2, what is the probability the car is actually behind door 3?"

To do this, we can <u>introduce 3 collections of events</u> (which are actually partitions according to defn 3.8).

In each case $i \in \{1,2,3\}$:

- F_i : You initially select door i.
- G_i : Monty reveals a goat behind door i.
- C_i : The car is behind door i.

Then we are looking for

$$P(C_3 | F_1 \cap G_2)$$

We can easily compute $P(G_2 | F_1 \cap C_3)$
If the car is behind door 3 and we picked door 1, Monty has to reveal a goat behind door 2, so

$$P(G_2 | F_1 \cap C_3) = 1$$

Similarly we find

$$P(G_2 | F_1 \cap C_2) = 0$$

$$P(G_2 | F_1 \cap C_1) = 1/2$$

Bayes theorem with everything conditional on our door choice:

$$P(C_3 | G_2 \cap F_1) = \frac{P(G_2 | C_3 \cap F_1) P(C_3 | F_1)}{P(G_2 | F_1)}$$

(by Thm/corollary given before)

Use law of total probability on denominator using partition $\{C_i \mid i \in 1,2,3\}$

$$P(G_2 \mid F_1) = P_{F_1}(G_2)$$

(partition thm on P_{F_1})

$$= P_{F_1}(G_2 \mid C_1) . P_{F_1}(C_1) + P_{F_1}(G_2 \mid C_2) . P_{F_1}(C_2) + P_{F_1}(G_2 \mid C_3) . P_{F_1}(C_3)$$

$$= P(G_2 \mid F_1 \cap C_1) . P(C_1 \mid F_1) + P(G_2 \mid C_2 \cap F_1) . P(C_2 \mid F_1)$$